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Tugas PDP (F) – 21 Mei 2024

$$xU_{xx} - yU_{yy} + \frac{1}{2}(U_x - U_y) = 0$$

a). Tentukan domain persamaan elliptic dan hiperbolik.

※ Bagian utama : $aU_{xx} + 2bU_{xy} + cU_{yy} = 0$

$\Rightarrow a = x \,;\; b = 0 \,;\; c = -y$

▶ Domain Elliptic

$$b^2 - ac < 0$$
$$0^2 - (x)(-y) < 0$$
$$xy < 0$$

Sehingga, akan memenuhi domain elliptic, saat $x > 0$ dan $y < 0$ atau $x < 0$ dan $y > 0$.

▶ Domain Hiperbolik

$$b^2 - ac > 0$$
$$0^2 - x(-y) > 0$$
$$xy > 0$$

Sehingga, akan memenuhi domain hiperbolik, saat $x > 0$ dan $y > 0$ atau $x < 0$ dan $y < 0$

b). Temukan transformasi Kanonik yang sesuai

▶ Hiperbolik

$$\frac{dy}{dx} = \frac{b \pm \sqrt{b^2 - ac}}{a} = \frac{0 \pm \sqrt{0^2 - x(-y)}}{x} = \pm\frac{\sqrt{xy}}{x} = \pm\frac{\sqrt{y}}{\sqrt{x}}$$

$\cdot\!>$ $\dfrac{dy}{dx} = \dfrac{\sqrt{y}}{\sqrt{x}}$

$$\int \frac{1}{\sqrt{y}}\,dy = \int \frac{1}{\sqrt{x}}\,dx$$
$$2\sqrt{y} = 2\sqrt{x} + C$$
$$\sqrt{y} - \sqrt{x} = C_1$$

$\cdot\!>$ $\dfrac{dy}{dx} = -\dfrac{\sqrt{y}}{\sqrt{x}}$

$$\int \frac{1}{\sqrt{y}}\,dy = \int -\frac{1}{\sqrt{x}}\,dx$$
$$2\sqrt{y} = -2\sqrt{x} + C_2$$
$$C_2 = \sqrt{y} + \sqrt{x}$$

Diperoleh : $s(x,y) = \sqrt{y} - \sqrt{x}$

$r(x,y) = \sqrt{y} + \sqrt{x}$

fungsi $\vartheta(s,r) = u(x,y)$

$\cdot\!>$ $U_x = \vartheta_s s_x + \vartheta_r r_x = \vartheta_s\left(-\frac{1}{2\sqrt{x}}\right) + \vartheta_r\left(\frac{1}{2\sqrt{x}}\right) = \frac{1}{2\sqrt{x}}(\vartheta_r - \vartheta_s)$

$\cdot\!>$ $U_y = \vartheta_s s_y + \vartheta_r r_y = \vartheta_s\cdot\left(\frac{1}{2\sqrt{y}}\right) + \vartheta_r\left(\frac{1}{2\sqrt{y}}\right) = \frac{1}{2\sqrt{y}}(\vartheta_s + \vartheta_r)$

$\cdot\!>$ $U_{xx} = [\vartheta_{ss}s_x + \vartheta_{sr}r_x]s_x + \vartheta_s[s_{xx}] + [\vartheta_{rs}s_x + \vartheta_{rr}r_x]r_x + \vartheta_r[r_{xx}]$

$\Rightarrow V_{ss}s_x^2 + \vartheta_{sr}r_x s_x + \vartheta_s s_{xx} + \vartheta_{rs}s_x r_x + \vartheta_{rr}r_x^2 + \vartheta_r r_{xx}$

$= \vartheta_{ss}s_x^2 + 2\vartheta_{sr}r_x s_x + \vartheta_{rr}r_x^2 + \vartheta_s s_{xx} + \vartheta_r r_{xx}$

$$= v_{ss}\left[\frac{1}{4x}\right] + 2v_{sr}\left[-\frac{1}{4x}\right] + v_{rr}\left[\frac{1}{4x}\right] + v_s\left[\frac{1}{4x\sqrt{x}}\right] + v_r\left[-\frac{1}{4x\sqrt{x}}\right]$$

$$= \frac{1}{4x}\left[v_{ss} - 2v_{sr} + v_{rr} + \frac{v_s}{\sqrt{x}} - \frac{v_r}{\sqrt{x}}\right]$$

- $U_{yy} = \left[v_{ss} s_y + v_{sr} r_y\right] s_y + v_s s_{yy} + \left[v_{rs} s_y + v_{rr} r_y\right] r_y + v_r r_{yy}$

$$= v_{ss} s_y^2 + 2v_{rs} s_y r_y + v_{rr} r_y^2 + v_s s_{yy} + v_r r_{yy}$$

$$= v_{ss}\left[\frac{1}{4y}\right] + 2v_{rs}\left[\frac{1}{4y}\right] + v_{rr}\left[\frac{1}{4y}\right] + v_s\left[-\frac{1}{4y\sqrt{y}}\right] + v_r\left[-\frac{1}{4y\sqrt{y}}\right]$$

$$= \frac{1}{4y}\left[v_{ss} + 2v_{rs} + v_{rr} - \frac{v_s}{\sqrt{y}} - \frac{v_r}{\sqrt{y}}\right]$$

▶ Substitusikan U_x, U_y, U_{xx}, U_{yy} ke PD awal:

- $x U_{xx} - y U_{yy} + \frac{1}{2}(U_x - U_y) = 0$
- $x \cdot \frac{1}{4x}\left[v_{ss} - 2v_{sr} + v_{rr} + \frac{v_s}{\sqrt{x}} - \frac{v_r}{\sqrt{x}}\right]$

$$- y \cdot \frac{1}{4y}\left[v_{ss} + 2v_{rs} + v_{rr} - \frac{v_s}{\sqrt{y}} - \frac{v_r}{\sqrt{y}}\right] + \frac{1}{2}\left[\frac{1}{2\sqrt{x}}(v_r - v_s) - \frac{1}{2\sqrt{y}}(v_r + v_s)\right] = 0$$

$$= \frac{1}{4}\left[\cancel{v_{ss}} - 2v_{sr} + \cancel{v_{rr}} + \cancel{\frac{v_s}{\sqrt{x}}} - \cancel{\frac{v_r}{\sqrt{x}}}\right] - \frac{1}{4}\left[\cancel{v_{ss}} + 2v_{rs} + \cancel{v_{rr}} - \cancel{\frac{v_s}{\sqrt{y}}} - \cancel{\frac{v_r}{\sqrt{y}}}\right] + \frac{1}{4}\left[\cancel{\frac{v_r}{\sqrt{x}}} - \cancel{\frac{v_s}{\sqrt{x}}} - \cancel{\frac{v_r}{\sqrt{y}}} - \cancel{\frac{v_s}{\sqrt{y}}}\right] = 0$$

$$= \frac{1}{4}\left[-2v_{sr} - 2v_{sr}\right] = 0$$

$$= v_{rs} = 0$$

$$\frac{\partial^2 v}{\partial r \partial s} = 0$$

$$\frac{\partial}{\partial r}\left[\frac{\partial v}{\partial s}\right] = 0$$

$$\int \partial\left[\frac{\partial v}{\partial s}\right] = \int 0 \, \partial r$$

$$= \frac{\partial v}{\partial s} = f(s)$$

$$\int \partial v = \int f(s)\, \partial s$$

$$v = F(s) + g(r)$$

Sehingga, transformasi Kanonik yang sesuai adalah $v_{rs} = 0$ dan PUPD nya adalah $v = F(s) + g(r)$ //

▶ Elliptic

$$\frac{dy}{dx} = \frac{b \pm i\sqrt{ac - b^2}}{a}$$

$$= \pm\frac{i\sqrt{x(-y)}}{x}$$

$$\frac{dy}{dx} = \frac{\pm i\sqrt{(-y)}}{\sqrt{x}}$$

$$\int \frac{1}{(\sqrt{-y})} dy = \int \pm i \frac{1}{\sqrt{x}} dx$$

$$2(\sqrt{-y}) = \pm 2\sqrt{x}\, i + C$$

$$(\sqrt{-y}) = \pm\sqrt{x}\, i + C$$

$$C = (\sqrt{-y}) \pm i\sqrt{x}$$

Sehingga : $r(x,y) = \sqrt{(-y)}$ $\quad s(x,y) = \sqrt{x}$

$r_x = 0 \qquad r_y = \frac{1}{2\sqrt{(-y)}}$

$s_x = \frac{1}{2\sqrt{x}} \qquad s_y = 0$

$r_{yy} = -\frac{1}{4(-y)\sqrt{y}} \qquad s_{xx} = -\frac{1}{4x\sqrt{x}}$

Karena diketahui $v(r,s) = u(x,y)$, maka :

$\Rightarrow U_x = v_r r_x + v_s s_x$

$$= v_r(0) + v_s\left(\frac{1}{2\sqrt{x}}\right)$$

$$= \frac{v_s}{2\sqrt{x}}$$

$\Rightarrow U_y = v_r r_y + v_s s_y$

$$= \frac{v_r}{2\sqrt{(-y)}}$$

$$\Rightarrow U_{xx} = \left[\vartheta_{sr} r_x + \vartheta_{ss} s_x\right] s_x + \left[\vartheta_s\right] s_{xx}$$

$$= \vartheta_r (0) + \vartheta_{ss} \cdot \frac{1}{2\sqrt{x}} \cdot \frac{1}{2\sqrt{x}} + \vartheta_s \cdot \left(-\frac{1}{4x\sqrt{x}}\right)$$

$$= \frac{\vartheta_{ss}}{4x} - \frac{\vartheta_s}{4x\sqrt{x}} = \frac{1}{4x}\left(\vartheta_{ss} - \frac{\vartheta_s}{\sqrt{x}}\right)$$

$$\Rightarrow U_{yy} = \left[\vartheta_{rr} r_y + \vartheta_{rs} s_y\right] r_y + \vartheta_r \cdot r_{yy}$$

$$= -\left[\left[\vartheta_{rr} \cdot \frac{1}{2\sqrt{(-y)}} + \vartheta_{rs}(0)\right]\frac{1}{2\sqrt{(-y)}} + \vartheta_r \cdot \left(-\frac{1}{4(-y)\sqrt{-y}}\right)\right]$$

$$= -1\left[\frac{\vartheta_{rr}}{4(-y)} - \frac{\vartheta_r}{4(-y)\sqrt{-y}}\right] = \frac{1}{4y}\left[\vartheta_{rr} + \frac{\vartheta_r}{4\sqrt{-y}}\right]$$

Substitusi ke PD Awal

$$x U_{xx} - y U_{yy} + \frac{1}{2}\left(U_x - U_y\right) = 0$$

$$x \frac{1}{4x}\left[\vartheta_{ss} - \frac{\vartheta_s}{\sqrt{x}}\right] - y \cdot \frac{1}{4y}\left[\vartheta_{rr} + \frac{\vartheta_r}{4\sqrt{-y}}\right] + \frac{1}{4}\left[\frac{\vartheta_s}{\sqrt{x}} + \frac{\vartheta_r}{\sqrt{(-y)}}\right] = 0$$

$$= \frac{1}{4}\left[\vartheta_{ss} - \cancel{\frac{\vartheta_s}{\sqrt{x}}} - \vartheta_{rr} - \cancel{\frac{\vartheta_r}{\sqrt{-y}}} + \cancel{\frac{\vartheta_s}{\sqrt{x}}} + \cancel{\frac{\vartheta_r}{\sqrt{-y}}}\right] = 0$$

$$\vartheta_{ss} - \vartheta_{rr} = 0$$

Sehingga, bentuk kanoniknya $\boxed{\vartheta_{ss} - \vartheta_{rr} = 0}$